which will happen if and only if $n > 1$ and $d > 0$ are such that its discriminant

$$(n-1)^2n^2d^2 - \frac{2}{3}(n-1)n^2(2n-1)d^2 + 4n$$

is non-negative and this is equivalent to saying $(n-1)n(n+1)d^2 \leq 12$. Consequently the greatest common difference is

$$d = \sqrt{\frac{12}{(n-1)n(n+1)}}\,,$$

and the first term of the progression is the solution of the quadratic above with this value of d, namely

$$x_1 = -\sqrt{\frac{3(n-1)}{n(n+1)}}.$$

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; ROY BARBARA, Lebanese University, Fanar, Lebanon; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One incomplete solution was received.

3572. [2010 : 397, 399] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\left(\sum_{\text{cyclic}} \frac{ab}{c+ab}\right) + \frac{1}{4}\prod_{\text{cyclic}}\left(\frac{a+\sqrt{ab}}{a+b}\right) \geq 1\,.$$

Composite of similar solutions by Arkady Alt, San Jose, CA, USA; Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina; and Albert Stadler, Herrliberg, Switzerland.

Note first that

$$\sum_{\text{cyclic}} \frac{ab}{c+ab} = \sum_{\text{cyclic}} \frac{ab}{c(a+b+c)+ab} = \sum_{\text{cyclic}} \frac{ab}{(c+a)(c+b)}$$
$$= \frac{1}{(a+b)(b+c)(c+a)}\sum_{\text{cyclic}} ab(a+b)\,.$$

Hence the given inequality is equivalent to

$$4\sum_{\text{cyclic}} ab(a+b) + \prod_{\text{cyclic}}(a+\sqrt{ab}) \geq 4(a+b)(b+c)(c+a),$$

or

$$\prod_{\text{cyclic}}(a+\sqrt{ab}) \geq 8abc\,.$$

By the AM-GM Inequality, we have

$$\prod_{\text{cyclic}} (a + \sqrt{ab}) = \sqrt{abc} \prod_{\text{cyclic}} (\sqrt{a} + \sqrt{b})$$
$$\geq 8\sqrt{abc}\sqrt{\sqrt{a}\sqrt{b}\sqrt{b}\sqrt{c}\sqrt{c}\sqrt{a}} = 8abc,$$

so our proof is complete. Clearly, equality holds if and only if $a = b = c = \frac{1}{3}$.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; OLIVER GEUPEL, Brühl, NRW, Germany; JOE HOWARD, Portales, NM, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; and the proposer.

3574. [2010 : 398, 400, 548, 550] *Proposed by Michel Bataille, Rouen, France.*

Let x, y, and z be real numbers such that $x + y + z = 0$. Prove that

$$\sum_{\text{cyclic}} \cosh x \;\leq\; \sum_{\text{cyclic}} \cosh^2\left(\frac{x-y}{2}\right) \;\leq\; 1 + 2\sum_{\text{cyclic}} \cosh x\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

Let $a := e^x, b = e^y, c = e^z$. Then $a, b, c > 0$ and $abc = e^{x+y+z} = 1$. Let $s := a + b + c, p := ab + ac + bc$. Then

$$\sum_{cyc} \cosh(x) = \frac{1}{2}\sum_{cyc}(a + bc) = \frac{s+p}{2}\,.$$

Let's observe that

$$\cosh\left(\frac{x-y}{2}\right) = \frac{e^{\frac{x-y}{2}} + e^{\frac{y-x}{2}}}{2} = \frac{1}{2}\left(\frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}}\right)$$
$$= \frac{a+b}{2\sqrt{ab}} = \frac{(a+b)\sqrt{c}}{2}\,. \qquad (1)$$

Thus

$$\sum_{cyc} \cosh^2\left(\frac{x-y}{2}\right) = \frac{1}{4}\sum_{cyc}(a+b)^2 c = \frac{1}{4}\sum_{cyc} a^2c + b^2c + 2 = \frac{3 + sp}{4}\,.$$

Also

$$\prod_{cyc} \cosh(x) = \prod_{cyc} \frac{a+bc}{2} = \prod_{cyc} \frac{a^2+1}{2a}$$
$$= \frac{1}{8}\prod_{cyc}(a^2+1) = \frac{2 + p^2 + s^2 - 2p - 2s}{8}\,.$$